Example Integral:

Find $$\int (2t-7)e^{3t+4}\,dt$$

parts $u = 2t-7 \Rightarrow du = 2\,dt$

$dv = e^{3t+4} \Rightarrow v = \frac{1}{3}e^{3t+4}$

$$= (2t-7)\frac{1}{3}e^{3t+4} - \int \frac{1}{3}e^{3t+4}\cdot 2\,dt$$

$$= \frac{1}{3}(2t-7)e^{3t+4} - \frac{2}{3}\int e^{3t+4}\,dt$$

$$= \frac{1}{3}(2t-7)e^{3t+4} - \frac{2}{3}\frac{1}{3}e^{3t+4} + C$$

$$= e^{3t+4}\left(\frac{2}{3}t + \left[\frac{-7\cdot 3}{3\cdot 3} - \frac{2}{9}\right]\right) + C$$

$$= \boxed{e^{3t+4}\left[\frac{2}{3}t - \frac{23}{9}\right] + C}$$

Example

$$\int \frac{1}{\sqrt{17-y^2}}\,dy = \int \frac{1}{\sqrt{(\sqrt{17})^2-y^2}}\,dy$$

Let $y = \sqrt{17}\,u \Rightarrow dy = \sqrt{17}\,du$ $\Rightarrow u = \frac{y}{\sqrt{17}}$

$$= \int \frac{1}{\sqrt{(\sqrt{17})^2-(\sqrt{17}u)^2}}\sqrt{17}\,du = \int \frac{1\cdot\sqrt{17}}{\sqrt{17(1-u^2)}}\,du$$

$$= \int \frac{1 \cdot \cancel{\sqrt{17}}}{\cancel{\sqrt{17}}\sqrt{1-u^2}}\,du = \arcsin(u) + C$$

$$= \boxed{\arcsin\left(\frac{y}{\sqrt{17}}\right) + C}$$

Theme of today: funkier substitutions

- subbing in trig functions.

→ to get rid of square roots.

$s^2 + c^2 = 1$

$t^2 + 1 = \sec^2$

We will utilize $\cos^2(\theta) = 1 - \sin^2(\theta)$

$$\tan^2(\theta) = \sec^2(\theta) - 1$$

$$\sec^2(\theta) = \tan^2(\theta) + 1$$

If we have $(1-u^2)^{3/2}$ or $\frac{1}{\sqrt{u^2+1}}$ in our integral, we can let $u = \sin\theta$ or $u = \tan\theta$

If we have $(u^2-1)^{-5/2}$ ← let $u = \sec\theta$.

Example. Find $\int \frac{1}{\sqrt{p^2+4}}\,dp$.

Let's try $p = 2\tan\theta \Rightarrow p^2 + 4 = 4\tan^2\theta + 4$

$$= 4(\tan^2\theta + 1)$$

$$= 4\sec^2\theta.$$

$$dp = 2\sec^2\theta \, d\theta$$

$$\int \frac{1}{\sqrt{p^2+4}}\,dp = \int \frac{2\sec^2(\theta)\,d\theta}{\sqrt{\underbrace{4\tan^2\theta+4}_{4\sec^2\theta}}}$$

$$= \int \frac{2\sec^2\theta}{2\sec\theta}\,d\theta = \int \sec\theta \, d\theta$$

$$\int \frac{(\sec\theta)(\sec\theta+\tan\theta)}{(\sec\theta+\tan(\theta))}\,d\theta = \int \frac{\sec^2\theta + \tan\theta\sec\theta}{\sec\theta+\tan\theta}\,d\theta$$

let $u = \sec\theta + \tan\theta$

$du = (\sec\theta\tan\theta + \sec^2\theta)\,d\theta$

$$= \int \frac{du}{u} = \ln|u| + C$$

$$= \ln|\sec\theta + \tan\theta| + C$$

Recall $p = 2\tan\theta \Rightarrow \tan\theta = \frac{p}{2}$

$\sec^2\theta = \tan^2\theta + 1 = \frac{p^2}{4} + 1$

$\sec\theta = \sqrt{\frac{p^2}{4}+1}$

$$= \boxed{\ln\left|\sqrt{\frac{p^2}{4}+1} + \frac{p}{2}\right| + C}$$

$\boxed{\text{Example}}$ $\int (5-4x-x^2)^{3/2}\,dx$

Vaguely looks like $1-u^2$

To make this look like that, we need to <u>complete the square</u>

divide by 2, square

$$5-4x-x^2 = 5-(x^2+4x+4)+4$$

$$= 9-(x+2)^2$$

$$\int (9-(x+2)^2)^{3/2}\,dx$$

Let $x+2 = 3\sin\theta \Rightarrow dx = 3\cos\theta\,d\theta$

$$= \int \big(\underbrace{9-(3\sin\theta)^2}\big)^{3/2} 3\cos\theta\,d\theta$$

$$9-9\sin^2\theta$$

$$9(1-\sin^2\theta)$$

$$9\cdot\cos^2\theta$$

$$= \int (9\cos^2\theta)^{3/2}\cdot 3\cos\theta\,d\theta$$

$$9^{3/2} = (9^{\frac{1}{2}})^3 = 3^3 = 27$$

$(A^B)^C = A^{B\cdot C}$

$$= \int 9^{3/2} (\cos^2\theta)^{3/2} \cdot 3\cos\theta \, d\theta$$

$$= \int 27 \cdot \cos^3(\theta) \cdot 3 \cdot \cos\theta \, d\theta$$

$$= 81 \int \cos^4\theta \, d\theta$$

$$= 81 \int (\cos^2\theta)(\cos^2\theta) \, d\theta$$

$$= 81 \int \left(\tfrac{1}{2} + \tfrac{1}{2}\cos(2\theta)\right)\left(\tfrac{1}{2} + \tfrac{1}{2}\cos(2\theta)\right) d\theta$$

$$= \frac{81}{4} \int (1+\cos(2\theta))(1+\cos(2\theta)) \, d\theta$$

$$1 + 2\cos(2\theta) + \cos^2(2\theta)$$

$$= \frac{81}{4} \int \left(1 + 2\cos(2\theta) + \cos^2(2\theta)\right) d\theta$$

$$= \frac{81}{4} \int \left(1 + 2\cos(2\theta) + \tfrac{1}{2} + \tfrac{1}{2}\cos(4\theta)\right) d\theta$$

$$= \frac{81}{4} \int \left(\tfrac{3}{2} + 2\cos(2\theta) + \tfrac{1}{2}\cos(4\theta)\right) d\theta$$

$$= \frac{243}{8}\theta + \frac{81}{4}\sin(2\theta) + \frac{81}{32}\sin(4\theta) + C$$

$x+2 = 3\sin\theta \Rightarrow \frac{x+2}{3} = \sin\theta \Rightarrow \theta = \arcsin\left(\frac{x+2}{3}\right)$

$\sin(2\theta) = 2\sin\theta \underbrace{\cos\theta}_{\sqrt{1-\sin^2\theta}} = 2\left(\frac{x+2}{3}\right) \cdot \sqrt{1-\left(\frac{x+2}{3}\right)^2}$

$$\sin(4\theta) = 2\sin(2\theta)\cos(2\theta)$$

$$= 4\sin\theta\underbrace{\cos\theta}_{\sqrt{1-\sin^2\theta}}\left(1-2\sin^2\theta\right)$$

$$= 4\left(\frac{x+2}{3}\right)\sqrt{1-\left(\frac{x+2}{3}\right)^2}\left(1-2\left(\frac{x+2}{3}\right)^2\right)$$

Our integral

$$= \boxed{\frac{243}{8}\arcsin\left(\frac{x+2}{3}\right)+\frac{81}{2}\left(\frac{x+2}{3}\right)\sqrt{1-\left(\frac{x+2}{3}\right)^2} + \frac{81}{8}\left(\frac{x+2}{3}\right)\sqrt{1-\left(\frac{x+2}{3}\right)^2}\left(1-2\left(\frac{x+2}{3}\right)^2\right)+C}$$